Symmetric Homology of Algebras

S. Ault and Z. Fiedorowicz

August 11, 2007

Abstract

In this note, we outline the general development of a theory of symmetric homology of algebras, an analog of cyclic homology where the cyclic groups are replaced by symmetric groups. This theory is developed using the framework of crossed simplicial groups and the homological algebra of module-valued functors. The symmetric homology of group algebras is related to stable homotopy theory. Two spectral sequences for computing symmetric homology are constructed. The relation to cyclic homology is discussed and some conjectures and questions towards further work are discussed.

2000 MSC: 16E40, 55P45, 55S12

Symmetric homology is the analog of cyclic homology, where the cyclic groups are replaced by symmetric groups. The second author and Loday [6] developed the notion of *crossed simplicial group* as a framework for making this idea precise.

Definition 1 A *crossed simplicial group* is a category ΔG whose objects are the sets $[n] = \{0, 1, 2, \dots, n\}$ for $n \geq 0$, which contains the simplicial category Δ, and such that any morphism $[m] \to [n]$ factors uniquely as

$$[m] \xrightarrow{\cong} [m] \xrightarrow{\gamma} [n],$$

where γ is a morphism in Δ. The collection of groups $\{G_n = Aut_{\Delta G}([n])^{op}\}_{n\geq 0}$ are called the *underlying groups* of ΔG. The commutation relations implicit in ΔG endow $\{G_n\}_{n\geq 0}$ with the structure of a simplicial set (but not necessarily the structure of a simplicial group).

The standard well-known example of a crossed simplicial group is ΔC, whose underlying groups are the cyclic groups $\{\mathbf{Z}_{n+1}\}_{n\geq 0}$. Less well-known is ΔS, whose underlying groups are the symmetric groups $\{\Sigma_{n+1}\}_{n\geq 0}$. The description of ΔS given in [6] is difficult to work with. A much nicer construction of this category is due to Pirashvili [11].

Definition 2 The category ΔS has objects $[n] = \{0, 1, 2, \ldots, n\}$. A morphism $f : [m] \to [n]$ is a set function together with a specification of a total order on the point preimages $\{f^{-1}(i)\}_{0\leq i\leq n}$. Composition of morphisms $[m] \xrightarrow{f} [n] \xrightarrow{g} [p]$ is given by specifying the order on $(gf)^{-1}(i) = \amalg_{j\in g^{-1}(i)} f^{-1}(j)$, as the block ordering specified by the ordering on $g^{-1}(i)$ and then within each block by the ordering specified on each $f^{-1}(j)$. Any morphism $f : [m] \to [n]$ decomposes uniquely as the permutation on $[m]$ specified by $\amalg_{0\leq i\leq n} f^{-1}(i)$ followed by an order preserving function $[m] \to [n]$, which is thus in Δ. The cyclic category ΔC is the evident subcategory of ΔS.

Remark Pirashvili's construction is a special case of a more general construction due to May and Thomason [10]. This construction associates to any topological operad $\{\mathcal{C}(k)\}_{n\geq 0}$ a topological category $\widehat{\mathcal{C}}$ together with a functor from this category to finite sets such that the inverse image of any function $f : [m] \to [n]$ is the space $\prod_{i=0}^{n} \mathcal{C}(|f^{-1}(i)|)$. Composition in $\widehat{\mathcal{C}}$ is defined using the composition of the operad. May and Thomason refer to $\widehat{\mathcal{C}}$ as the *category of operators* associated to $\mathcal{C}$. They were interested in the case of an E_∞ operad, but their construction evidently works for any operad. The category of operators associated to the discrete A_∞ operad $\mathcal{A}ss$, which parametrizes monoid structures, is precisely Pirashvili's construction of ΔS.

Now given any small category $\mathcal{C}$ and any commutative ring k, one can define homological algebra of covariant and contravariant functors $F : \mathcal{C} \longrightarrow k$-modules. The simplest way to describe this is to consider the ring $k[\mathcal{C}]$, which is the free k-module generated by all the morphisms in $\mathcal{C}$. Multiplication is defined on this basis by composition if the morphisms are composable and 0 otherwise. A covariant functor $F : \mathcal{C} \longrightarrow k$-modules is then exactly the same thing as a left $k[\mathcal{C}]$-module structure on $\bigoplus_{C\in \mathrm{Obj}(\mathcal{C})} F(C)$. Similarly contravariant functors correspond to right $k[\mathcal{C}]$-modules (equivalently, left $k[\mathcal{C}^{\mathrm{op}}]$-modules).

One then defines for contravariant F and covariant G,

$$\mathrm{Tor}_*^{\mathcal{C}}(F, G) = \mathrm{Tor}_*^{k[\mathcal{C}]}\left(\bigoplus_{C \in \mathrm{Obj}(\mathcal{C})} F(C), \bigoplus_{C \in \mathrm{Obj}(\mathcal{C})} G(C)\right)$$

(There are some small technicalities that need to be checked, as the ring $k[\mathcal{C}]$ does not have a multiplicative unit if $\mathcal{C}$ has infinitely many objects. But it does have local units which are sufficient to carry this out.)
If A is a k-algebra, then the cyclic bar construction defines a functor $B^{cyc}A : \Delta C^{op} \longrightarrow k$-modules, and cyclic homology can be defined as

$$HC_*(A) = \mathrm{Tor}_*^{\Delta C}(B^{cyc}A, \underline{k}),$$

where $\underline{k} : \Delta C \longrightarrow k$-modules denotes the trivial functor which takes every object to k and every morphism to the identity.
However the results of [6] were discouraging as to the prospect of an analogous definition of symmetric homology. First of all, the cyclic bar construction does not extend to a functor $\Delta S^{op} \longrightarrow k$-modules. Secondly it was shown that for any functor $F : \Delta S^{op} \longrightarrow k$-modules, $\mathrm{Tor}^{\Delta S}(F, \underline{k})$ is just the homology of the underlying simplicial module of F, given by restricting F to Δ^{op}.
Subsequently, the second author [5] noticed that the cyclic bar construction extends not to a contravariant functor on ΔS but to a covariant functor.

Definition 3 The *symmetric bar construction* is the functor $B^{sym}A : \Delta S \longrightarrow k$-modules which takes the object $[n]$ to the $(n+1)$-fold tensor product $A^{\otimes n+1}$ of A with itself over k. If $f : [m] \to [n]$ is a morphism in ΔS, then $B^{sym}(f)$ takes $a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_m$ to $b_0 \otimes b_1 \otimes b_2 \otimes \ldots \otimes b_n$, where $b_i = \prod_{j \in f^{-1}(i)} a_j$, where the product is taken in the order specified on $f^{-1}(i)$.

The cyclic bar construction can be identified with the composite

$$\Delta C^{op} \overset{D}{\cong} \Delta C \subset \Delta S \xrightarrow{B^{sym}} k\text{-modules},$$

where D is a suitable duality isomorphism.
This now allows us to define symmetric homology as

Definition 4 $HS_*(A) = \mathrm{Tor}_*^{\Delta S}(\underline{k}, B^{sym}A)$, where $\underline{k} : \Delta S^{op} \longrightarrow k$-modules denotes the trivial functor which takes every object to k and every morphism to the identity.

One can use the standard bar resolution of $\underline{k}$ to compute $HS_*(A)$ as the homology of the simplicial abelian group $L_*(A)$, where

$$L_p(A) = \bigoplus k\left[[m_0] \xrightarrow{f_1} [m_1] \xrightarrow{f_2} [m_2] \xrightarrow{f_3} \dots \xrightarrow{f_p} [m_p]\right] \otimes A^{\otimes m_0+1}.$$

Here the direct sum ranges over all composable chains of morphisms in ΔS of length p. The 0-th face consists of deleting f_1 and acting on $A^{\otimes m_0+1}$ via $B^{sym}(f_1)$. The p-th face consists of dropping f_p. The other faces are given by composing f_{i+1} with f_i. The degeneracies consist of inserting identities.
If $A = k[M]$ is a monoid ring, then $HS_*(A)$ has a simpler description: it is the homology with k-coefficients of the nerve of the category whose set of objects is the disjoint union $\amalg_{n\geq 0} M^{n+1}$. A morphism from $(m_0, m_1, m_2, \dots, m_p) \in M^{p+1}$ to $(m'_0, m'_1, m'_2, \dots, m'_q) \in M^{q+1}$ is a morphism $f : [p] \to [q]$ in ΔS, such that $m'_i = \prod_{j \in f^{-1}(i)} m_j$. In the special case when $M = J(X_+)$ is the free monoid on a generating set X (for which we have $A = k[M] = T(X)$, the tensor algebra on X) we have the following result.

Theorem 1 *$HS_*(T(X)) = HS_*(k[J(X_+)]) = H_*(C_\infty(X_+); k)$, where C_∞ denotes the monad associated to the little ∞-cubes operad ([8], [4]).*

We may replace C_∞ above by the monad associated to any E_∞ operad. In particular it is preferable to use the monad associated to the operad $\mathcal{D}$ (see Theorem 3 below).
If the monoid is a group G, we have the following result.

Theorem 2 *$HS_*(k[G]) = H_*(\Omega\Omega^\infty S^\infty(BG); k)$*

The special case when G is free abelian of rank n is of particular interest. In this case the group ring is the ring of Laurent polynomials in n indeterminates. On the other hand BG is the n-torus which stably splits into a wedge of spheres. Thus we obtain

Corollary 1 $HS_*\left(k[t_1^\pm, t_2^\pm, \dots, t_n^\pm]\right) = H_*(\Omega\Omega^\infty S^\infty(\bigvee_{i=1}^n \bigvee_{j=1}^{n!/(i!(n-i)!)} S^i); k)$

$$= H_*(\prod_{i=1}^n \prod_{j=1}^{n!/(i!(n-i)!)} \Omega^\infty S^\infty(S^{i-1}); k)$$

Since the symmetric homology of the group completion of a commutative monoid is the group completion of the symmetric homology of the monoid, a natural conjecture would be

Conjecture 1 $HS_*(k[t_1, t_2, \ldots, t_n]) =$

$$H_*\big(\prod_{i=1}^n C_\infty(S^0) \times \prod_{i=2}^n \prod_{j=1}^{n!/(i!(n-i)!)} \Omega^\infty S^\infty(S^{i-1}); k\big)$$

In the case $n = 1$, this conjecture is a special case of Theorem 1.
The E_∞ structure visible in the above examples is a general phenomenon present in $HS_*(A)$ for any algebra. In order to make this precise, we need to enlarge the category ΔS by adding an initial object $[-1]$. Call the resulting enlarged category ΔS_+, and let $L_*^+(A)$ be the resulting enlarged bar complex. Then ΔS_+ is a strict symmetric monoidal category (with the monoidal structure given by the coproduct) and we have

Theorem 3 *(a)* $HS_*(A) = H_*(L_*^+(A))$
(b) $L_*^+(A)$ *is an* E_∞ *chain complex with respect to the action of the* E_∞ *operad* $\mathcal{D}$.
(c) If $k = Z_p$, p *a prime, then* $HS_*(A)$ *is equipped with Dyer-Lashof homology operations.*

The E_∞ chain operad which acts on $L_*^+(A)$ is the chain operad associated to the operad $\mathcal{D} = \{E\Sigma_n\}_{n\geq 0}$ which acts on strict symmetric monoidal (a.k.a. permutative) categories [9]. This operad, in its simplicial form, is known as the Barratt-Eccles operad.
The following related result is joint work with Tomas Barros.

Theorem 4 *(a) The chain complex* $L_*^+(A)$ *is equipped with a Smith filtration ([3], [14]). The* n*-stage of this filtration is an* E_n *chain complex.*
(b) If $A = k[G]$ *is a group ring, then the homology of the* n*-stage of the Smith filtration on* $L_*^+(A)$ *is isomorphic to* $H_*(\Omega^n S^{n-1}(BG); k)$.

While the chain complex $L_*(A)$ fortuitously lends itself to computations of $HS_*(A)$ in the special cases of the monoid rings of free monoids and group rings, it is much too unwieldy for computations in general, as it is infinite dimensional in each degree. As a first step in obtaining a more tractable chain complex, we have

Proposition 1 *If* A *is equipped with an augmentation* $A \to k$ *and* I *denotes the augmentation ideal, then the inclusion* $L_*^{epi}(A) \subset L_*(A)$ *is a chain homotopy equivalence, where*

$$L_p^{epi}(A) = \bigoplus k\left[[m_0] \overset{f_1}{\twoheadrightarrow} [m_1] \overset{f_2}{\twoheadrightarrow} [m_2] \overset{f_3}{\twoheadrightarrow} \ldots \overset{f_p}{\twoheadrightarrow} [m_p]\right] \otimes B_{m_0}^{sym} I,$$

for $p > 0$, where the f_i are required to be epimorphisms. Here

$$B^{sym}_m I = \begin{cases} A & \text{if } m = 0 \\ I^{\otimes m+1} & \text{if } m > 0 \end{cases}$$

Thus $HS_*(A) = H_*(L^{epi}_*(A))$.

The chain complex $L^{epi}_*(A)$ in turn can be filtered in a couple of ways, giving rise to spectral sequences for computing $HS_*(A)$. The simplest such spectral sequence arises by filtering $L^{epi}_*(A)$ by the number of jumps: the n-th filtration of $L^{epi}_*(A)$ consists of chains where at most n of the $[m_{i-1}] \xrightarrow{f_i} [m_i]$ are strict (i.e. $m_{i-1} > m_i$). We obtain

Theorem 5 *If A is equipped with an augmentation with augmentation ideal I, then there is a first quadrant spectral sequence converging to $HS_*(A)$ with*

$$E^1_{p,q} = \bigoplus_{m_0 > m_1 > m_2 > \ldots > m_p \geq 0} H_q\left(\Sigma^{op}_{m_p+1}; B^{sym}_{m_0} I \otimes k\left[\prod_{i=1}^{p} Epi_\Delta([m_{i-1}], [m_i])\right]\right)$$

Here $\mathrm{Epi}_\Delta([m], [n])$ denotes the set of epimorphisms in Δ between $[m]$ and $[n]$. The group homology is defined with respect to the group right action of Σ_{m_p+1} given by the isomorphism

$$\begin{aligned} B^{sym}_{m_0} I \otimes k\left[\prod_{i=1}^{p} \mathrm{Epi}_\Delta([m_{i-1}], [m_i])\right] \\ \cong B^{sym}_{m_0} I \otimes_{k[\Sigma_{m_0+1}]} k\left[\mathrm{Epi}_{\Delta S}([m_0], [m_1])\right] \otimes_{k[\Sigma_{m_1+1}]} k\left[\mathrm{Epi}_{\Delta S}([m_1], [m_2])\right] \\ \otimes_{k[\Sigma_{m_2+1}]} \ldots \otimes_{k[\Sigma_{m_{p-1}+1}]} k\left[\mathrm{Epi}_{\Delta S}([m_{p-1}], [m_p])\right] \end{aligned}$$

.

Here, the $B^{sym}_{m_0} I$ component comes before the chain of morphisms because we are viewing it as a *right* $k\Sigma_{m_0+1}$-module rather than a *left* $k\Sigma^{\mathrm{op}}_{m_0+1}$-module. The differential $E^1_{p,q} \to E^1_{p-1,q}$ is an alternating sum of faces. The 0-th face takes

$$a_0 \otimes a_1 \otimes \ldots a_{m_0} \otimes \left\{[m_0] \stackrel{f_1}{\twoheadrightarrow} [m_1] \stackrel{f_2}{\twoheadrightarrow} [m_2] \stackrel{f_3}{\twoheadrightarrow} \ldots \stackrel{f_p}{\twoheadrightarrow} [m_p]\right\}$$

to

$$B^{sym}(f_1)(a_0 \otimes a_1 \otimes \ldots a_{m_0}) \otimes \left\{[m_1] \stackrel{f_2}{\twoheadrightarrow} [m_2] \stackrel{f_3}{\twoheadrightarrow} \ldots \stackrel{f_p}{\twoheadrightarrow} [m_p]\right\}.$$

The middle faces compose consecutive arrows. The last face is induced by $f_p^* : \Sigma_{m_p+1} \to \Sigma_{m_{p-1}+1}$, which is part of the simplicial structure on the underlying groups $\{\Sigma_{n+1}\}_{n\geq 0}$ of ΔS.
Now, since the differential of $L_*^{epi}(A)$ reduces the filtration degree by at most one, it can be shown that the differentials $E^r_{p,q} \to E^r_{p-r,q+r-1}$ must be trivial for $r \geq 2$. Hence, the spectral sequence collapses at the E^2 term.
This spectral sequence is still not very computationally useful as the E^1-term is infinitely generated in each degree. A better spectral sequence is obtained by filtering $L_*^{epi}(A)$ as follows:

$$F_m L_p^{epi}(A) = \bigoplus_{m_0 \leq m} B_{m_0}^{sym} I \otimes k \left[[m_0] \overset{f_1}{\twoheadrightarrow} [m_1] \overset{f_2}{\twoheadrightarrow} [m_2] \overset{f_3}{\twoheadrightarrow} \dots \overset{f_p}{\twoheadrightarrow} [m_p] \right],$$

We obtain the following result.

Theorem 6 *If A is equipped with an augmentation whose augmentation ideal I is a free k-module with basis X, then there is a spectral sequence converging strongly to $HS_*(A)$ with*

$$E^1_{p,q} = \bigoplus_{\overline{u} \in X^{p+1}/\Sigma_{p+1}} \widetilde{H}_{p+q}(EG_{\overline{u}} \ltimes_{G_{\overline{u}}} N\mathcal{S}_p / N\mathcal{S}'_p; k)$$

Here $G_{\overline{u}}$ is the isotropy subgroup of the orbit $\overline{u} \in X^{p+1}/\Sigma_{p+1}$. $N\mathcal{S}_p$ is the nerve of the category $\mathcal{S}_p$, which is defined as follows. Let $\{z_0, z_1, z_2, \dots\}$ be a countable set of indeterminates. First we define a larger category $\tilde{\mathcal{S}}_p$. The objects of $\tilde{\mathcal{S}}_p$ are formal tensor products $Z_0 \otimes Z_1 \otimes \dots \otimes Z_r$ where each Z_i is a formal (nonempty) product of the indeterminates $\{z_0, z_1, \dots, z_p\}$ so that $Z_0 Z_1 \dots Z_r = z_{\sigma(0)} z_{\sigma(1)} \dots z_{\sigma(p)}$ for some $\sigma \in \Sigma_{p+1}$. In other words each z_i, $i = 0, 1, 2, \dots, p$ occurs once and only once as a factor in exactly one of the products Z_j, $j = 0, 1, 2, \dots, r$. There is precisely one morphism in $\tilde{\mathcal{S}}_p$ $Z_0 \otimes Z_1 \otimes \dots \otimes Z_r \longrightarrow Y_0 \otimes Y_1 \otimes \dots \otimes Y_s$ iff each Y_i is a product of some of the monomials Z_j's. We then take $\mathcal{S}_p$ to be a skeletal subcategory of $\tilde{\mathcal{S}}_p$. $\mathcal{S}_p$ is a poset. The nerve $N\mathcal{S}_p$ is contractible, since $\mathcal{S}_p$ contains the initial object $z_0 \otimes z_1 \otimes \dots \otimes z_p$. We then take $\mathcal{S}'_p$ to be the subposet obtained from $\mathcal{S}_p$ by deleting the initial object. Thus the quotient $N\mathcal{S}_p/N\mathcal{S}'_p$ has the same homotopy type as the suspension of $N\mathcal{S}'_p$. The symmetric group Σ_{p+1} acts on $\mathcal{S}_p$ by permuting the generators $\{z_0, z_1, z_2, \dots, z_p\}$. This induces an action on $N\mathcal{S}_p/N\mathcal{S}'_p$. The differential $E^1_{p,q} \longrightarrow E^1_{p-1,q}$ is induced by the 0-th face map in $N\mathcal{S}_p$.

Thus a fundamental problem in computing symmetric homology is to determine the homotopy type of the spaces $N\mathcal{S}_p/N\mathcal{S}'_p$ and to analyze the actions of the symmetric groups on these spaces. If k is a field of characterisitic 0, just knowing the rational homology of these spaces and the action of the symmetric groups on the homology would suffice to determine the E^1-term of the spectral sequence of Theorem 6. However the chain complex of the simplicial nerve of $N\mathcal{S}_p/N\mathcal{S}'_p$ is too bulky to permit computations except for very small values of p.
One can apply a similar technique, as is used to derive Theorem 5, to the nerve of the nonskeletal category $\tilde{\mathcal{S}}_p$ to obtain a much smaller chain complex $Sym_*^{(p)}$, which computes the homology of $N\mathcal{S}_p/N\mathcal{S}'_p$. The group of i-chains $Sym_i^{(p)}$ is the free abelian group on the objects of $\tilde{\mathcal{S}}_p$ having the form $Z_0 \otimes Z_1 \otimes Z_2 \otimes \ldots \otimes Z_{p-i}$, modded out by the equivalence relation generated by

$$\begin{aligned} &Z_0 \otimes Z_1 \otimes \ldots \otimes Z_{j-1} \otimes Z_j \otimes \ldots \otimes Z_{p-i} \\ &= \ (-1)^{(|Z_{j-1}|+1)(|Z_j|+1)} Z_0 \otimes Z_1 \otimes \ldots \otimes Z_j \otimes Z_{j-1} \otimes \ldots \otimes Z_{p-i} \end{aligned}$$

where $|Z|$ denotes the length of the product. The boundary map in $Sym_*^{(p)}$ is an alternating sum of faces, where each face consists of splitting a product Z_j into a tensor product $Z'_j \otimes Z''_j$ (so that $Z_j = Z'_j Z''_j$ and the faces are ordered according to the position of the new $\otimes$. For example

$$\partial(z_2 z_0 z_3 \otimes z_1 z_4) = z_2 \otimes z_0 z_3 \otimes z_1 z_4 - z_2 z_0 \otimes z_3 \otimes z_1 z_4 + z_2 z_0 z_3 \otimes z_1 \otimes z_4$$

The action of Σ_{p+1} on $Sym_*^{(p)}$ is induced by permutation of the generators $\{z_0, z_1, z_2, \ldots, z_p\}$.
The direct sum $\bigoplus_{p\geq 0} Sym_*^{(p)}$ forms a bigraded differential algebra, where $Sym_i^{(p)}$ is assigned bigrading $(p+1, i)$. The product

$$\boxtimes : Sym_i^{(p)} \otimes Sym_j^{(q)} \longrightarrow Sym_{i+j}^{(p+q+1)}$$

is given by $Y \boxtimes Z = Y \otimes Z'$, where Z' is obained from Z by replacing each generator z_r by z_{r+p+1} for $r = 0, 1, 2, \ldots, q$. The product is related to the boundary map by the the relation

$$\partial(Y \boxtimes Z) = \partial(Y) \boxtimes Z + (-1)^i Y \boxtimes \partial(Z),$$

when Y has bigrade $(p+1, i)$. Thus there is an induced map in homology:

$$\boxtimes : H_i(Sym_*^{(p)}) \otimes H_j(Sym_*^{(q)}) \longrightarrow H_{i+j}(Sym_*^{(p+q+1)})$$

The product $\boxtimes$, both on the chain level and the homology level, is not strictly skew commutative, but rather skew commutative in a twisted sense:

$$Y \boxtimes Z = (-1)^{ij} \sigma Z \boxtimes Y$$

where σ is the permutation which sends $0, 1, 2, \ldots q$ to $p+1, p+2, \ldots, p+q+1$ and $q+1, q+2, \ldots, p+q+1$ to $0, 1, 2, \ldots, p$ in an order preserving way.
It is easy to compute the top degree homology groups. Let

$$\begin{aligned} b_p &= z_0 z_1 z_2 \ldots z_p + (-1)^p z_1 z_2 \ldots z_p z_0 + (-1)^{2p} z_2 z_3 \ldots z_p z_0 z_1 \\ &\quad + \ldots + (-1)^{p^2} z_p z_0 z_1 z_2 \ldots z_{p-1}. \end{aligned}$$

Then b_p is a cycle and thus a homology class. As a $\mathbf{Z}[\Sigma_{p+1}]$-module, $H_p(Sym_*^{(p)})$ is generated by b_p and as a representation $H_p(Sym_*^{(p)})$ is either the sign representation on $\mathbf{Z}_{p+1}$ (if p is odd) or the trivial representation on $\mathbf{Z}_{p+1}$ (if p is even), induced up to Σ_{p+1}. Thus $H_p(Sym_*^{(p)})$ is free abelian of rank $p!$.
We summarize our calculations so far below.

Theorem 7 *For $p = 0, 1, 2, 3, 4, 5, 6, 7$ $H_*(Sym_*^{(p)})$ are free abelian and have the following Poincaré polynomials:*

$$p_0(t) = 1,\ p_1(t) = t,\ p_2(t) = t + 2t^2,\ p_3(t) = 7t^2 + 6t^3,$$
$$p_4(t) = 43t^3 + 24t^4,\ p_5(t) = t^3 + 272t^4 + 120t^5,$$
$$p_6(t) = 36t^4 + 1847t^5 + 720t^6,$$
$$p_7(t) = 829t^5 + 13710t^6 + 5040t^7$$

Ideally we would like to describe generators and relations for $\bigoplus_{p \geq 0} H_*(Sym_*^{(p)})$ with respect to the module structures over the group rings of the symmetric groups and the $\boxtimes$ product. The calculations summarized above show that besides the generators b_i constructed above, there are additional generators in $H_3(Sym_*^{(4)})$, $H_4(Sym_*^{(5)})$, $H_5(Sym_*^{(6)})$, and $H_6(Sym_*^{(7)})$. For now we only have very limited understanding of these additional generators or of the relations between the generators. For instance we have the following relation in $H_2(Sym_*^{(3)})$

$$b_1 \boxtimes b_1 = (1 + [0312] + [1230])\, b_2 \boxtimes b_0,$$

where $[abcd]$ stands for the permutation $0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d$. The calculations also establish that $N\mathcal{S}_p / N\mathcal{S}'_p$ has the homotopy type of a wedge of spheres for $p \leq 6$.

In recent work [15] Vrećica and Živaljević have connected $Sym_*^{(p)}$ to a certain well-studied class of geometric complexes, known as chessboard complexes [16]. Using this they have shown that $Sym_*^{(p)}$ is $\lfloor \frac{2}{3}(p-1) \rfloor$-connected. This result implies that the connectivity of the spaces $N\mathcal{S}_p/N\mathcal{S}'_p$ is an increasing function of p, hence the spectral sequence of Theorem 6 converges in the strong sense. Indeed, for $m > \frac{3}{2}(i+1)$, there is an isomorphism

$$H_i(F_m L_p^{epi}(A)) \xrightarrow{\cong} HS_i(A)$$

If we denote by $\Delta^{(m)}S$ the full subcategory of ΔS consisting of the objects $[0], [1], \ldots, [m]$, then it follows that for $m > \frac{3}{2}(i+1)$,

$$HS_i(A) = Tor_i^{\Delta^{(m)}S}(\underline{k}, B^{sym}A).$$

Observe that $k\left[\Delta^{(m)}S\right]$ is a finite-dimensional unital ring, hence if A is finitely generated over a Noetherian ground ring k, then the increasing connectivity of the spaces $N\mathcal{S}_p/N\mathcal{S}'_p$ implies that $HS_*(A)$ is finite dimensional over k in each degree. In the case when $A = k[G]$ is the group ring of a finite group, this also follows from Theorem 2, the Atiyah-Hirzebruch spectral sequence for stable homotopy theory and Serre $\mathcal{C}$-theory.
Some questions are suggested by our partial computations of $H_*(Sym_*^{(p)})$: Is it true that the homology is always torsion-free? Or might it even be true that the spaces $N\mathcal{S}_p/N\mathcal{S}'_p$ are always wedges of spheres? Can the Vrećica and Živaljević connectivity result be improved to $H_i(Sym_*^{(p)}) = 0$ for $i \leq p - r$, where

$$r = \left\lfloor \frac{\sqrt{8p+9}-1}{2} \right\rfloor \approx \sqrt{2p}?$$

If this were true, this would be the best possible connectivity result, since the sign representation of Σ_{p+1} has nontrivial multiplicity in all $H_i(Sym_*^{(p)})$ for $p - r < i \leq p$. The computed multiplicities of the trivial representations are also consistent with this hypothesis.

We also have the following results on symmetric homology in degrees 0 and 1.

Proposition 2 *(a) $HS_i(A)$ for $i = 0, 1$ is the homology of the following partial chain complex*

$$0 \longleftarrow A \xleftarrow{\partial_1} A \otimes A \otimes A \xleftarrow{\partial_2} (A \otimes A \otimes A \otimes A) \oplus A$$

where

$$\partial_1(a \otimes b \otimes c) = abc - cba$$

$$\partial_2(a\otimes b\otimes c\otimes d) = ab\otimes c\otimes d + d\otimes ca\otimes b + bca\otimes 1\otimes d + d\otimes bc\otimes a, \quad \partial_2(a) = 1\otimes a\otimes 1$$

(b) $HS_0(A) = A/[A,A]$ *is the symmetrization of* A *(as an algebra).*

We also have an elaboration of Theorem 1, which describes symmetric homology as the homology of the E_∞ symmetrization of an algebra. The idea is to simplicially resolve the algebra by tensor algebras, then in each simplicial degree replace the tensor algebra by a free E_∞ chain algebra on the same generators, and finally to take the double complex of the resulting simplicial chain algebra. A more precise formulation is

Theorem 8 $HS_*(A) = H_*(B(D,T,A))$, *where* $B(D,T,A)$ *is the 2-sided bar construction,* T *is the functor which takes a* k*-module to the tensor algebra on that module,* D *is the monad which takes a* k*-module to the free* $\mathcal{D}$ *chain algebra over that module (where* $\mathcal{D}$ *is the same operad as in Theorem 3), and* $B(D,T,A)$ *is converted from a simplicial chain complex to a double complex.*

Finally we briefly discuss the relation between symmetric homology and cyclic homology. The relation between the cyclic bar construction and the symmetric bar construction, discussed above, leads to a natural map

$$HC_*(A) \longrightarrow HS_*(A).$$

The same analysis as in Theorems 5 and 6 can be carried out for cyclic homology. The cyclic analog of $N\mathcal{S}'_p$ can be identified as a simplicial complex with the barycentric subdivision of the boundary of a p-simplex. The cyclic group acts on this $p-1$ sphere by cyclicly permuting the vertices of the simplex. The cyclic analog of $N\mathcal{S}_p/N\mathcal{S}'_p$ is homotopy equivalent to the suspension of this and is thus a p-sphere. One can then combine the cyclic analog of Theorem 6 with the Serre spectral sequence for computing the homology of the resulting half-smash products to obtain the standard spectral sequence for cyclic homology.

We can use the partial chain complex of Proposition 3 and an analogous one for cyclic homology (c.f. [7], page 59) to describe the map $HC_i(A) \longrightarrow HS_i(A)$ for $i = 0, 1$. These maps are induced by the following partial chain map:

$$\begin{tikzcd}
0 & A \arrow[l] \arrow[d, "\mathrm{id}"] & A \otimes A \arrow[l, "ab-ba"'] \arrow[d, "a\otimes b\otimes 1"] & A^{\otimes 3} \oplus A \arrow[l, "\partial_2^C"'] \arrow[d, "f"] \\
0 & A \arrow[l] & A^{\otimes 3} \arrow[l, "abc-cba"'] & A^{\otimes 4} \oplus A \arrow[l, "\partial_2^S"']
\end{tikzcd}$$

The map ∂_2^C takes $a \otimes b \otimes c \in A^{\otimes 3}$ to $ab \otimes c - a \otimes bc + ca \otimes b$, and takes $a \in A$ to $1 \otimes a - a \otimes 1$. The map ∂_2^S is the map ∂_2 from Proposition 3. f is a map that is defined on the first summand by

$$\begin{aligned} a \otimes b \otimes c \quad \mapsto \quad & a \otimes b \otimes c \otimes 1 - 1 \otimes a \otimes bc \otimes 1 + 1 \otimes ca \otimes b \otimes 1 \\ & +1 \otimes 1 \otimes abc \otimes 1 - b \otimes ca \otimes 1 \otimes 1 - 2abc - cab \end{aligned}$$

and on the second summand by

$$a \mapsto 4a - 1 \otimes 1 \otimes a \otimes 1$$

The map $HC_0(A) \longrightarrow HS_0(A)$ is the quotient map which takes the quotient of A by the k-module generated by all commutators onto the quotient of A by the ideal generated by all commutators.
In a similar vein, Pirashvili and Richter (c.f. [12] and [13]) have shown that

$$HC_*(A) = Tor_*^{\Delta S}(\underline{b}, B^{sym}A),$$

where $\underline{b}$ is the contravariant functor on ΔS which is the cokernel of $d_0 - d_1 : P_1 \longrightarrow P_0$, where $P_i(-) = k[hom_{\Delta S}(-, [i])]$ and $d_0 - d_1$ induces the commutator map $a \otimes b \mapsto ab - ba$ on $B^{Sym}A$. Thus the natural map $HC_*(A) \longrightarrow HS_*(A)$ is induced by the unique natural transformation $\underline{b} \to \underline{k}$. Moreover the proof of Proposition 3 shows that $\underline{k}$ is the cokernel of $f - g : P_2 \longrightarrow P_0$, where $f - g$ induces $a \otimes b \otimes c \mapsto abc - cba$ on $B^{sym}A$, and the partial chain map above is induced by a map from a partial projective resolution of $\underline{b}$ over $k[\Delta S]$ to a partial projective resolution of $\underline{k}$ over $k[\Delta S]$.
The Vrećica and Živaljević connectivity theorem, discussed above, implies that there is a projective resolution of $\underline{k}$ which in degree i is a finite direct sum of the projective modules P_m with $m \leq \frac{3}{2}(i + 1)$.

References

[1] R. Aboughazi, Les groupes simpliciaux croisés, Ph. D. thesis, Strasbourg, 1987.

[2] S. Ault, On the symmetric homology of algebras, Ph. D. thesis, The Ohio State University, in progress, `http://www.math.ohio-state.edu/~ault/Thesis.pdf`.

[3] C. Berger, Combinatorial models for real configuration spaces and E_n operads, Contemp. Math. 202 (1997), 37–52.

[4] F. Cohen, T. Lada, J. P. May, The Homology of Iterated Loop Spaces, Springer Lecture Notes in Math. 533, Springer-Verlag, Berlin, 1972.

[5] Z. Fiedorowicz, The symmetric bar construction, preprint, `http://www.math.ohio-state.edu/~fiedorow/symbar.ps.gz`.

[6] Z. Fiedorowicz and J-L. Loday. Crossed simplicial groups and their associated homology. Trans. Amer. Math. Soc., 326 (1991), 57-87.

[7] J-L. Loday, Cyclic Homology, Springer-Verlag, Berlin, 1998.

[8] J. P. May, The Geometry of Iterated Loop Spaces, Springer Lecture Notes in Math. 271, Berlin, 1972.

[9] J. P. May, E_∞ spaces, group completions, and permutative categories, in New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), 61-93, 1974.

[10] J. P. May and R. W. Thomason, The uniqueness of infinite loop space machines, Topology 17 (1978), 205-224.

[11] T. Pirashvili, On the PROP corresponding to bialgebras, Cah. Topol. Géom. Différ. Catég. 43 (2002), 221-239.

[12] T. Pirashvili and B. Richter, Hochschild and cyclic homology via functor homology, K-Theory 25 (2002), 39-49.

[13] J. Słomińska, Decompositions of the category of noncommutative sets and Hochschild and cyclic homology, Cent. Eur. J. Math. 1 (2003), 327-331.

[14] J. H. Smith, Simplicial group models for $\Omega^n S^n(X)$, Israel J. Math. 66 (1989), 330-350.

[15] S. T. Vrećica and R. R. Živaljević, Cycle-free chessboard complexes and symmetric homology of algebras, preprint, http://arxiv.org/abs/0710.5252.

[16] M. L. Wachs, Topology of matching, chessboard, and general bounded degree graph complexes, Algebra Universalis 49 (2003), 345-385.

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210-1174, USA
Email: ault@math.ohio-state.edu, fiedorow@math.ohio-state.edu